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Quark mass dependence of the one-loop three-gluon vertex in arbitrary dimension

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Abstract: The one-loop off-shell massive quark contribution to the three-gluon vertex is calculated in an arbitrary space-time dimension. The results for all relevant on-shell and symmetric limits are obtained directly from the general off-shell results. The analytic structure of the results for the relevant massive scalar integrals is also discussed.

Keywords: QCD, NLO Computations.

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JHEP08(2001)050

Contents

1. Introduction

The three-gluon vertex is the basic object responsible for the non-abelian nature of Quantum Chromodynamics [1]. Perturbative corrections to gluonic vertices are very important in real physical calculations, such as multijet production at the hadron colliders (see e.g. [2] and references therein). At the present high level of accuracy, one needs to perform not only calculations with on-shell external particles, there are also contributions where general off-shell results are needed.

Several special cases for the one-loop three-gluon vertex have been known already for a couple of decades. Previous studies of the three-gluon vertex have mainly been carried out with massless quarks, or with no quarks at all.[1] Around 1980, Celmaster, Gonsalves, Pascual and Tarrach studied the one-loop three-gluon vertex with massless quarks in the symmetric limit, $p_1^2 = p_2^2 = p_3^2$, mainly for the purpose of comparing different renormalization schemes [3, 4]. The results for the case when

[1] Here and below, we mainly discuss the results in covariant gauges.

two gluons are on shell have been given by Nowak, Praszałowicz and Słomiński [5]. In the pure gluodynamics (leaving out the quark loops), Ball and Chiu considered the off-shell case in the Feynman gauge [6]. Brandt and Frenkel presented results for the infrared-singular parts with one and two on-shell gluons in an arbitrary covariant gauge [7].

More recently, general one-loop results for the three-gluon vertex, in an arbitrary covariant gauge and dimension, have been presented in [8] (see also [9] for a brief review). Table 1 of [8] gives an overview of the results for the one-loop three-gluon vertex obtained in the preceding papers. The results of [8] have contributed towards covering all remaining "white spots" in Table 1, with only one restriction: the case of massless quarks was considered.

The purpose of the present paper is to extend the work started in [8] and complete the study of the one-loop-order three-gluon vertex, by considering massive quarks in the quark-loop contributions, for an arbitrary value of the space-time dimension. We note that these contributions do not depend on the gauge parameter at one loop. Thereby, we would allow for non-zero quark masses in all configurations listed in Table 1 of [8], including the most general off-shell case. Taking into account that the corresponding investigation of the quark-gluon vertex with massive quarks is given in [10], and remembering that the one-loop ghost-gluon vertex [8] does not involve any quark contributions, one can see that this is indeed the last step, in addition to refs. [8, 10], needed to complete the most general study of the one-loop three-point vertices in QCD.

Such general results for the one-loop vertices can be useful in the evaluation of two-loop (or higher) order corrections. In particular, they can be used as "blocks" in evaluating higher-order corrections in QCD. Presenting results in arbitrary dimension n, we can obtain further terms of the expansion in $\varepsilon = (4-n)/2$, in the framework of dimensional regularization [11]. Moreover, we can derive results for all on-shell limits of interest directly from the general ones. From arbitrary dimension one can go over to two and three dimensions which are also investigated in the context of QCD (see, e.g., in [12]). Finally, we would like to mention recent progress in the lattice calculations related to the QCD vertices (see, e.g., in refs. [13]).

We also discuss the analytic structure of the results for the massive scalar integrals which appear in the calculations, using the geometrical approach of ref. [14]. Worth noting is that scalar integrals of this type occur not only in the three-gluon coupling but also in one-loop corrections to $H\gamma\gamma$, $HZ\gamma$, ZZH, W^+W^-H and some other vertices (see, e.g., in [15, 16]).

The rest of the paper is organized as follows. In section 2, we give the notation for the three-gluon vertex, present the relevant Ward-Slavnov-Taylor identity, and describe the tensor decomposition. In section 3, we list the most general off-shell results, and the corresponding on-shell limits for the vertex. The scalar integrals are discussed in section 4. Conclusions and a summary are given in section 5.

JHEP08(2001)050

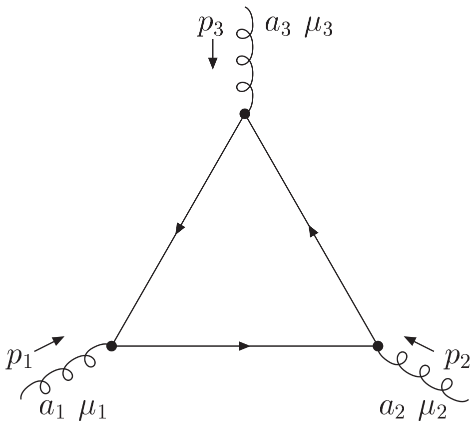

Figure 1: Notations used for the three-gluon vertex.

2. Preliminaries

2.1 Decomposition of the three-gluon vertex

Relevant notation for the three-gluon vertex,

$$\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) \equiv -\mathrm{i}gf^{a_1a_2a_3}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\,, \tag{2.1}$$

is given in figure 1. We note that all gluon momenta are ingoing, $p_1+p_2+p_3=0$. There are actually two diagrams, the fermion lines may be oriented either way. Because of the color factors, Furry's theorem does not apply, and the two diagrams add to produce a factor of two.

For the off-shell three-gluon vertex, we adopt the well-known decomposition proposed by Ball and Chiu [6],[2]

$$\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = {} & A(p_1^2,p_2^2;p_3^2)\; g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + B(p_1^2,p_2^2;p_3^2)\; g_{\mu_1\mu_2}(p_1+p_2)_{\mu_3} - \\
& -C(p_1^2,p_2^2;p_3^2)\left[(p_1p_2)g_{\mu_1\mu_2} - p_{1\mu_2}p_{2\mu_1}\right](p_1-p_2)_{\mu_3} + \\
& +\frac{1}{3}\,S(p_1^2,p_2^2,p_3^2)\left(p_{1\mu_3}p_{2\mu_1}p_{3\mu_2} + p_{1\mu_2}p_{2\mu_3}p_{3\mu_1}\right) + \\
& +F(p_1^2,p_2^2;p_3^2)\left[(p_1p_2)g_{\mu_1\mu_2} - p_{1\mu_2}p_{2\mu_1}\right]\times \\
& \times\left[p_{1\mu_3}(p_2p_3) - p_{2\mu_3}(p_1p_3)\right] + H(p_1^2,p_2^2,p_3^2)\times \\
& \times\Big\{-g_{\mu_1\mu_2}\left[p_{1\mu_3}(p_2p_3) - p_{2\mu_3}(p_1p_3)\right] + \\
& \quad + \frac{1}{3}\left(p_{1\mu_3}p_{2\mu_1}p_{3\mu_2} - p_{1\mu_2}p_{2\mu_3}p_{3\mu_1}\right)\Big\} + \\
& +\{\text{ cyclic permutations of } (p_1,\mu_1),(p_2,\mu_2),(p_3,\mu_3)\ \}\,.
\end{aligned} \tag{2.2}$$

Here, the A, C and F functions are symmetric in the first two arguments, the H function is totally symmetric, the B function is antisymmetric in the first two

[2] Another general decomposition of the three-gluon vertex was considered in ref. [17].

JHEP08(2001)050

arguments, while the S function is antisymmetric with respect to interchange of any pair of arguments.[3] Note that the contributions containing the F and H functions are totally transverse, i.e., they give zero when contracted with any of $p_{1\mu_1}$, $p_{2\mu_2}$ or $p_{3\mu_3}$.

2.2 Basic integrals

For the basic integrals, we follow the notation of refs. [8, 10] (see also in [18]). The scalar three-point integrals with equal masses associated with all three internal lines are defined as

$$J_3(\nu_1,\nu_2,\nu_3) \equiv \int \frac{\mathrm{d}^n q}{[(p_2-q)^2-m^2]^{\nu_1}\,[(p_1+q)^2-m^2]^{\nu_2}\,[q^2-m^2]^{\nu_3}}\,. \tag{2.3}$$

Here and henceforth, the causal prescription is understood, $1/p^2 \leftrightarrow 1/(p^2+\mathrm{i}0)$.

The following massive integrals are involved in the three-gluon vertex calculation:

$$\begin{aligned} J_3(1,1,1) &= \mathrm{i}\,\pi^{n/2}\,\eta\,\varphi_3\,, \\ J_3(0,1,1) &= \mathrm{i}\,\pi^{n/2}\,\eta\,\kappa_{2,1}\,, \\ J_3(1,0,1) &= \mathrm{i}\,\pi^{n/2}\,\eta\,\kappa_{2,2}\,, \\ J_3(1,1,0) &= \mathrm{i}\,\pi^{n/2}\,\eta\,\kappa_{2,3}\,, \\ J_3(0,0,1) &= \mathrm{i}\,\pi^{n/2}\,\eta\,m^2\,\widetilde{\kappa}\,, \end{aligned} \tag{2.4}$$

where the overall factor η (depending on $n = 4-2\varepsilon$) is defined by

$$\eta \equiv \frac{\Gamma^2(n/2-1)}{\Gamma(n-3)}\,\Gamma(3-n/2) = \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\Gamma(1+\varepsilon). \tag{2.5}$$

In eqs. (2.4), $\varphi_3 \equiv \varphi_3(p_1^2,p_2^2,p_3^2;m)$ is the non-trivial function associated with the scalar triangle integral. The subscript "3" indicates that all three internal lines are massive. This three-point function is discussed in more detail in section 4. For the two-point integrals we introduce the functions

$$\kappa_2(p_l^2;m) \equiv \kappa_{2,l}\,, \tag{2.6}$$

where p_l $(l = 1,2,3)$ is the external momentum of the two-point function, whereas the subscript "2" shows that both internal propagators are massive. The relevant information about this two-point function can be found in [10, appendix C1]. Finally, $\widetilde{\kappa}$ corresponds to the tadpole contribution,

$$\widetilde{\kappa} \equiv \widetilde{\kappa}(m^2) \equiv \frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)}\,\frac{1}{\varepsilon(1-\varepsilon)}\,(m^2)^{-\varepsilon}\,. \tag{2.7}$$

As a rule, we will omit the mass arguments in φ_3, κ_2 and $\widetilde{\kappa}$.

[3] In ref. [8] it was shown that the one-loop S function vanishes at the one-loop order. This is also the case for massive quarks (see below).

2.3 Ward-Slavnov-Taylor identity

The Ward-Slavnov-Taylor (WST) identity for the three-gluon vertex reads (see, e.g., in refs. [19, 6, 20])

$$
\begin{aligned}
p_3^{\mu_3}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = & -\left(g_{\mu_1}^{\mu_3}p_1^2 - p_{1\mu_1}p_1^{\mu_3}\right) J(p_1^2)G(p_3^2)\widetilde{\Gamma}_{\mu_3\mu_2}(p_1,p_3;p_2) + \\
& + \left(g_{\mu_2}^{\mu_3}p_2^2 - p_{2\mu_2}p_2^{\mu_3}\right) J(p_2^2)G(p_3^2)\widetilde{\Gamma}_{\mu_3\mu_1}(p_2,p_3;p_1)\,, \qquad (2.8)
\end{aligned}
$$

where $J(p^2)$ and $G(p^2)$ are scalar functions in the gluon polarization operator and the ghost self-energy, respectively:

$$
\begin{aligned}
\Pi_{\mu_1\mu_2}^{a_1a_2}(p) &= -\delta^{a_1a_2}\left(p^2 g_{\mu_1\mu_2} - p_{\mu_1}p_{\mu_2}\right) J(p^2), \\
\widetilde{\Pi}^{a_1a_2}(p^2) &= \delta^{a_1a_2}\frac{p^2}{G(p^2)}\,. \qquad (2.9)
\end{aligned}
$$

The ghost-gluon vertex is given by

$$
\widetilde{\Gamma}_{\mu_3}^{a_1a_2a_3}(p_1,p_2;p_3) \equiv -\mathrm{i}g\ f^{a_1a_2a_3}p_1^{\mu}\widetilde{\Gamma}_{\mu\mu_3}(p_1,p_2;p_3)\,, \qquad (2.10)
$$

where p_1 is the out-ghost momentum, p_2 is the in-ghost momentum, p_3 and μ_3 are the momentum and the Lorentz index of the gluon (all momenta are ingoing).

As in ref. [8], we will denote the zero-loop-order quantities as $X^{(0)}$ and the one-loop-order contributions as $X^{(1)}$, so that the perturbative expansion is

$$
X = X^{(0)} + X^{(1)} + \cdots\,.
$$

Moreover, at the one-loop level, we label the (gauge-dependent) gluon and ghost contributions and the (gauge-independent) quark-loop contribution by the superscripts "ξ" and "q", respectively:

$$
X^{(1)} = X^{(1,\xi)} + X^{(1,q)}\,. \qquad (2.11)
$$

The present paper deals only with quark-loop contributions, $X^{(1,q)}$. All relevant $X^{(1,\xi)}$ contributions are available in [8].

The lowest-order contributions to $J(p^2)$ and $G(p^2)$ are

$$
J^{(0)}(p^2) = G^{(0)}(p^2) = 1\,, \qquad (2.12)
$$

whereas the ghost-gluon vertex at the lowest order is

$$
\widetilde{\Gamma}_{\mu\mu_3}^{(0)}(p_1,p_2;p_3) = g_{\mu\mu_3}\,. \qquad (2.13)
$$

The massive quark-loop contribution, $J^{(1,q)}$, of the gluon polarization operator is

$$
J^{(1,q)}(p^2) = \frac{g^2\eta}{(4\pi)^{n/2}}\frac{2\hat{N}_fT_R}{(n-1)p^2}\left\{[(n-2)p^2+4m^2]\kappa_2(p^2) - 2(n-2)m^2\widetilde{\kappa}\right\}, \qquad (2.14)
$$

where $T_R = \frac{1}{8}\,\mathrm{Tr}(I) = \frac{1}{2}$ (if $\mathrm{Tr}(I) = 4$), with I being the "unity" in the space of Dirac matrices. Furthermore, $\hat{N}_f$ represents a sum over fermions of different masses m,

$$\hat{N}_f X(m) \equiv \sum_{i=1}^{N_f} X(m_i)\,. \tag{2.15}$$

In particular, if all fermions have the same mass then $\hat{N}_f \Rightarrow N_f$.

The quark-loop contribution to the WST identity can be obtained from eq. (2.8),

$$\begin{aligned} p_3^{\mu_3}\Gamma^{(1,q)}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = &- \left(g_{\mu_1\mu_2}p_1^2 - p_{1\mu_1}p_{1\mu_2}\right) J^{(1,q)}(p_1^2) + \\ &+ \left(g_{\mu_1\mu_2}p_2^2 - p_{2\mu_1}p_{2\mu_2}\right) J^{(1,q)}(p_2^2)\,, \end{aligned} \tag{2.16}$$

where the function $J^{(1,q)}$ is given in eq. (2.14), and we have also taken into account eq. (2.10).

3. Results for the quark-loop contributions

3.1 General off-shell case

The results for the quark-loop contributions to the three-gluon vertex can be written out in terms of the scalar functions (2.4). It is convenient to define the symmetric combinations:

$$\begin{aligned} \mathcal{Q} &\equiv -\frac{1}{2}(p_1^2+p_2^2+p_3^2) = (p_1p_2)+(p_1p_3)+(p_2p_3)\,, \\ \mathcal{K} &\equiv p_1^2p_2^2 - (p_1p_2)^2 = -\frac{1}{4}\lambda(p_1^2,p_2^2,p_3^2)\,, \end{aligned} \tag{3.1}$$

where $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx$ is the Källén function (for other forms of $\mathcal{K}$, see [8, eq. (3.2)]).

The general results quoted below have been obtained using the computer algebra package REDUCE [21] and standard techniques,[4] for expressing all integrals in terms of a few basic ones, eq. (2.4).

For general values of p_1^2, p_2^2 and p_3^2 we find the following results for the scalar functions from eq. (2.2):

$$\begin{aligned} A^{(1,q)}(p_1^2,p_2^2;p_3^2) = \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{1}{(n-1)}\Big[&(n-2)(\kappa_{2,1}+\kappa_{2,2}) + 4m^2\left(\frac{\kappa_{2,1}}{p_1^2}+\frac{\kappa_{2,2}}{p_2^2}\right) - \\ &- 2(n-2)m^2\frac{p_1^2+p_2^2}{p_1^2p_2^2}\widetilde{\kappa}\Big]\,, \end{aligned} \tag{3.2}$$

[4]See in ref. [8] for more details. We note that there is a misprint in [8, eq. (A20)] $+Z_{010}$ should read $-Z_{010}$.

$$
\begin{aligned}
B^{(1,q)}(p_1^2,p_2^2;p_3^2) &= \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{1}{(n-1)}\Big[(n-2)(\kappa_{2,1}-\kappa_{2,2})+4m^2\Big(\frac{\kappa_{2,1}}{p_1^2}-\frac{\kappa_{2,2}}{p_2^2}\Big)+ \\
&\qquad\qquad + 2(n-2)m^2\frac{p_1^2-p_2^2}{p_1^2p_2^2}\widetilde{\kappa}\Big]\,, && (3.3)\\
C^{(1,q)}(p_1^2,p_2^2;p_3^2) &= \frac{2}{p_1^2-p_2^2}B^{(1,q)}(p_1^2,p_2^2;p_3^2)\,, && (3.4)\\
S^{(1,q)}(p_1^2,p_2^2,p_3^2) &= 0\,, && (3.5)\\
F^{(1,q)}(p_1^2,p_2^2;p_3^2) &= -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{1}{(n-1)(n-2)\mathcal{K}^2}\times \\
&\times\Big\{2(n^2-1)\mathcal{K}^{-1}(p_1p_2)(p_1p_3)(p_2p_3)\times \\
&\quad\times[p_3^2(p_1p_2)\varphi_3+(p_1p_3)\kappa_{2,1}+(p_2p_3)\kappa_{2,2}+p_3^2\kappa_{2,3}]+ \\
&\quad+2(n-1)(n-3)p_3^2(p_1p_2)[(p_1p_2)\varphi_3+\kappa_{2,3}]+ \\
&\quad+2(n-1)(p_3^2-4m^2)\left[2\mathcal{K}+3p_3^2(p_1p_2)\right]\varphi_3+ \\
&\quad+4(n-2)(p_3^2-4m^2)\left[(p_1p_3)\kappa_{2,1}+(p_2p_3)\kappa_{2,2}+p_3^2\kappa_{2,3}\right]- \\
&\quad-2(3n-5)(p_1p_2)\left[(p_1p_3)\kappa_{2,1}+(p_2p_3)\kappa_{2,2}\right]+ \\
&\quad+\left[n(n-4)\mathcal{K}-2(n+1)p_3^2(p_1p_2)\right](\kappa_{2,1}+\kappa_{2,2})+2(n-2)\times \\
&\quad\times\frac{m^2\mathcal{K}}{p_1^2p_2^2}\left[(p_3^2-2p_1^2)\kappa_{2,1}+(p_3^2-2p_2^2)\kappa_{2,2}-2(n-2)(p_1p_2)\widetilde{\kappa}\right]+ \\
&\quad+(n-2)\mathcal{K}(p_1-p_2)^2\Big(n-2+2m^2\frac{p_1^2+p_2^2}{p_1^2p_2^2}\Big)\frac{\kappa_{2,1}-\kappa_{2,2}}{p_1^2-p_2^2}\Big\}, && (3.6)\\
H^{(1,q)}(p_1^2,p_2^2,p_3^2) &= -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{2}{(n-1)(n-2)\mathcal{K}^2}\times \\
&\times\Big\{(n^2-1)\mathcal{K}^{-1}(p_1p_2)(p_1p_3)(p_2p_3)\times \\
&\quad\times[(p_1p_2)(p_1p_3)(p_2p_3)\varphi_3+(p_1p_2)(p_1p_3)\kappa_{2,1}+ \\
&\quad+(p_1p_2)(p_2p_3)\kappa_{2,2}+(p_1p_3)(p_2p_3)\kappa_{2,3}]- \\
&\quad-3(n-1)(p_1p_2)(p_1p_3)(p_2p_3)\times \\
&\quad\times\left[(\mathcal{Q}+4m^2)\varphi_3+\kappa_{2,1}+\kappa_{2,2}+\kappa_{2,3}\right] \\
&\quad+(n-1)(n-2)\mathcal{K}^2\varphi_3-2(n-2)^2\mathcal{K}m^2\widetilde{\kappa}+ \\
&\quad+(n-2)(p_1^2-4m^2)[p_1^2(p_2p_3)+(p_1p_2)(p_1p_3)]\kappa_{2,1}+ \\
&\quad+(n-2)(p_2^2-4m^2)[p_2^2(p_1p_3)+(p_1p_2)(p_2p_3)]\kappa_{2,2}+ \\
&\quad+(n-2)(p_3^2-4m^2)[p_3^2(p_1p_2)+(p_1p_3)(p_2p_3)]\kappa_{2,3}\Big\}\,. && (3.7)
\end{aligned}
$$

We note that $A^{(1,q)}$, $B^{(1,q)}$ and $C^{(1,q)}$ do not depend on the third argument, p_3^2. The function $H^{(1,q)}(p_1^2,p_2^2,p_3^2)$ is explicitly symmetric with respect to all arguments. Once more, we would like to note that the S function vanishes at the one-loop level.

In the limit $n\to 4$ ($\varepsilon\to 0$), only the $A^{(1,q)}$ function has a $1/\varepsilon$ singularity of an

JHEP08(2001)050

ultraviolet origin,

$$\frac{g^2\eta}{(4\pi)^{2-\varepsilon}}\;\frac{4}{3}N_f T_R\left(\frac{1}{\varepsilon}+\cdots\right). \tag{3.8}$$

It can be renormalized by the corresponding renormalization factor, Z_1.

To consider the massless limit $m=0$ of these functions, we should put $\widetilde{\kappa}\Rightarrow 0$ (massless tadpole), $\varphi_3\Rightarrow\varphi_0\equiv\varphi$, $\kappa_{2,i}\Rightarrow\kappa_{0,i}\equiv\kappa_i$. In this way, we reproduce [8, eqs. (3.17)–(3.22)].

3.2 Symmetric case, $p_1^2=p_2^2=p_3^2\equiv p^2$

In the symmetric limit $p_1^2=p_2^2=p_3^2\equiv p^2$ $[(p_1p_2)=(p_1p_3)=(p_2p_3)=-\frac{1}{2}p^2]$, the three-gluon vertex has only three independent tensor structures [3],

$$\begin{aligned}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = {} & [g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}+g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1}+g_{\mu_3\mu_1}(p_3-p_1)_{\mu_2}]\,G_0(p^2) - \\ & -(p_2-p_3)_{\mu_1}(p_3-p_1)_{\mu_2}(p_1-p_2)_{\mu_3}G_1(p^2) + \\ & +\left(p_{2\mu_1}p_{3\mu_2}p_{1\mu_3}-p_{3\mu_1}p_{1\mu_2}p_{2\mu_3}\right)G_2(p^2)\,. \end{aligned}\tag{3.9}$$

The G_i functions are related to the scalar functions in eq. (2.2) via (see in [8])

$$\begin{aligned} G_1(p^2) &= C(p^2,p^2;p^2)+\frac{1}{2}p^2F(p^2,p^2;p^2)\,, \\ G_2(p^2) &= G_1(p^2)+H(p^2,p^2,p^2)\,, \\ G_0(p^2) &= A(p^2,p^2;p^2)+\frac{1}{2}p^2G_2(p^2)\,. \end{aligned}\tag{3.10}$$

Note that in the symmetric case the B and S functions vanish at any order, because they are antisymmetric.

The one-loop quark contributions to the G_i functions can be obtained directly from the general expressions (3.2)–(3.7), using the substitution (see [10, appendix C])

$$\begin{aligned}\left.\frac{\kappa_2(p_2^2)-\kappa_2(p_1^2)}{p_2^2-p_1^2}\right|_{p_1^2=p_2^2=p^2} &= \frac{\mathrm{d}\kappa_2(p^2)}{\mathrm{d}p^2} \\ &= \frac{1}{2p^2(p^2-4m^2)}\left\{[(n-4)p^2+4m^2]\kappa_2(p^2)-2(n-2)m^2\widetilde{\kappa}\right\}. \end{aligned}\tag{3.11}$$

In this way, we obtain

$$\begin{aligned} G_0^{(1,q)}(p^2) &= -\hat{N}_fT_R\frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{2}{9(n-2)}\Big\{2[(3n-8)p^2+6m^2]\varphi_{3\mathrm{s}}-3(3n-8)\kappa_2(p^2)\Big\}, \\ G_1^{(1,q)}(p^2) &= -\hat{N}_fT_R\frac{g^2\,\eta}{(4\pi)^{n/2}}\frac{4}{27p^2}\Big\{4p^2\varphi_{3\mathrm{s}}+3\frac{n-4}{n-1}\kappa_2(p^2)+ \\ &\qquad\qquad +\frac{18m^2}{(n-1)p^2}[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}]\Big\}, \end{aligned}$$

$$G_2^{(1,q)}(p^2) = -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{4}{9p^2}\Big\{ \frac{2}{n-2}[(3n-8)p^2+6m^2]\varphi_{3\mathrm{s}} - \\ - 3\frac{n-4}{(n-1)(n-2)}\kappa_2(p^2) + \\ + \frac{18m^2}{(n-1)p^2}[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}]\Big\}\,, \qquad (3.12)$$

where $\varphi_{3\mathrm{s}} \equiv \varphi_3(p^2,p^2,p^2)$ is the three-point function in the symmetric limit whose analytic properties are discussed in detail in section 4.2. In the massless quark limit we reproduce the corresponding results [8, (3.33)–(3.35)].

We can also consider the limit $p^2 \to 0$, when all external gluons are on shell. Since there are some p^2 in the denominators of the r.h.s.'s of eqs. (3.12), we should keep a few terms of the expansion[5] in p^2,

$$\kappa_2(p^2) = \frac{1}{2}(n-2)\widetilde{\kappa}\left[1-\frac{n-4}{12}\,\frac{p^2}{m^2}+\frac{(n-4)(n-6)}{240}\,\frac{(p^2)^2}{m^4}+\mathcal{O}\left(\frac{(p^2)^3}{m^6}\right)\right], \qquad (3.13)$$

$$\varphi_{3\mathrm{s}} = \varphi_3(p^2,p^2,p^2) = \frac{(n-4)(n-2)}{8m^2}\widetilde{\kappa}\left[1-\frac{n-6}{8}\,\frac{p^2}{m^2}+\mathcal{O}\left(\frac{(p^2)^2}{m^4}\right)\right]. \qquad (3.14)$$

In this way, we obtain

$$G_0^{(1,q)}(0) = \hat{N}_f T_R \frac{g^2\eta}{(4\pi)^{n/2}}\,\frac{2}{3}(n-2)\widetilde{\kappa}\,,$$
$$G_1^{(1,q)}(0) = -\hat{N}_f T_R \frac{g^2\eta}{(4\pi)^{n/2}}\,\frac{1}{15m^2}(n-4)(n-2)\widetilde{\kappa}\,,$$
$$G_2^{(1,q)}(0) = -\hat{N}_f T_R \frac{g^2\eta}{(4\pi)^{n/2}}\,\frac{17}{60m^2}(n-4)(n-2)\widetilde{\kappa}\,. \qquad (3.15)$$

Therefore, the three-gluon vertex is in the totally on-shell case given by eq. (3.9) with the functions (3.15).

3.3 One gluon on shell, $p_3^2=0$

In the limit $p_3^2=0$ $[(p_1p_2)=-\frac{1}{2}(p_1^2+p_2^2)]$, the tensor structures in eq. (2.2) remain unchanged. The results for the scalar functions of arguments $(p_1^2,p_2^2,0)$ can be obtained from the general expressions (3.2)–(3.7). In fact, the results (3.2)–(3.4) do not depend on p_3^2. Therefore, their form is not changed. In the expressions (3.6) and (3.7), we should put $\kappa_{2,3} \Rightarrow \frac{1}{2}(n-2)\widetilde{\kappa}(1-\frac{n-4}{12}\,\frac{p_3^2}{m^2})$ [cf. Eq. (3.13)] and

$$\varphi_3(p_1^2,p_2^2,0) = \frac{p_1^2\varphi_3(p_1^2,0,0)-p_2^2\varphi_3(0,p_2^2,0)}{p_1^2-p_2^2} \qquad (3.16)$$

[5] Any number of terms of the small-p^2 expansion of $\kappa_2(p^2)$ and $\varphi_3(p^2,p^2,p^2)$ can be obtained from [18, eqs. (17), (37) and (38)].

[see eq. (4.25) below]. In this way, we get

$$
\begin{aligned}
F^{(1,q)}(p_1^2,p_2^2;0) = {} & \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{4}{(n-1)(n-2)(p_1^2-p_2^2)^2} \times \\
& \times \Big[n(n-2)(\kappa_{2,1}+\kappa_{2,2}) - 16(n-1)m^2 \times \\
& \quad \times \frac{p_1^2\varphi_3(p_1^2,0,0)-p_2^2\varphi_3(0,p_2^2,0)}{p_1^2-p_2^2} - 4(n-2)m^2\left(\frac{\kappa_{2,1}}{p_1^2}+\frac{\kappa_{2,2}}{p_2^2}\right) - \\
& \quad - 4n\frac{p_1^2\kappa_{2,1}-p_2^2\kappa_{2,2}}{p_1^2-p_2^2} - 16(n-2)m^2\frac{\kappa_{2,1}-\kappa_{2,2}}{p_1^2-p_2^2} + \\
& \quad + 2(n-2)^2 m^2 \frac{p_1^2+p_2^2}{p_1^2p_2^2}\widetilde{\kappa}\Big] , \\
H^{(1,q)}(p_1^2,p_2^2,0) = {} & \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{(n-1)(n-2)(p_1^2-p_2^2)^3} \times \\
& \times \Big\{ 4(n-1)[(n-2)p_1^2p_2^2+6m^2(p_1^2+p_2^2)] \times \\
& \quad \times [p_1^2\varphi_3(p_1^2,0,0)-p_2^2\varphi_3(0,p_2^2,0)] + 2(n^2-1)(p_1^2+p_2^2)^2 \times \\
& \quad \times (\kappa_{2,1}-\kappa_{2,2}) + 6(n-1)[(p_1^2)^2-(p_2^2)^2](\kappa_{2,1}+\kappa_{2,2}) + \\
& \quad + 4(n-2)[(4m^2-p_1^2)(3p_1^2+p_2^2)\kappa_{2,1} - \\
& \qquad\qquad - (4m^2-p_2^2)(p_1^2+3p_2^2)\kappa_{2,2}] - \\
& \quad - (n-2)^2[(n-1)(p_1^2+p_2^2)+16m^2](p_1^2-p_2^2)\widetilde{\kappa}\Big\} .
\end{aligned} \tag{3.17}
$$

For the functions with permuted arguments $(0,p_1^2,p_2^2)$, we obtain

$$
\begin{aligned}
A^{(1,q)}(0,p_1^2;p_2^2) = {} & \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{1}{n-1}\Big\{\frac{1}{3}(n-2)[3\kappa_{2,1}+(n-1)\widetilde{\kappa}] + \\
& \qquad\qquad + 2\frac{m^2}{p_1^2}[2\kappa_{2,1}-(n-2)\widetilde{\kappa}]\Big\} , \\
B^{(1,q)}(0,p_1^2;p_2^2) = {} & -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{1}{n-1}\Big\{\frac{1}{3}(n-2)[3\kappa_{2,1}-(n-1)\widetilde{\kappa}] + \\
& \qquad\qquad + 2\frac{m^2}{p_1^2}[2\kappa_{2,1}-(n-2)\widetilde{\kappa}]\Big\} , \\
C^{(1,q)}(0,p_1^2;p_2^2) = {} & -\frac{2}{p_1^2}B^{(1,q)}(0,p_1^2;p_2^2) , \\
F^{(1,q)}(0,p_1^2;p_2^2) = {} & -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{4}{(n-1)(n-2)(p_1^2-p_2^2)^3} \times \\
& \times \Big\{ 4(n-1)[(n-2)p_1^2p_2^2+4m^2(p_1^2+2p_2^2)] \times \\
& \quad \times \frac{p_1^2\varphi_3(p_1^2,0,0)-p_2^2\varphi_3(0,p_2^2,0)}{p_1^2-p_2^2} + \\
& \quad + 8p_2^2[(n-1)^2p_1^2+2p_2^2+8(n-2)m^2]\frac{\kappa_{2,1}-\kappa_{2,2}}{p_1^2-p_2^2} +
\end{aligned}
$$

$$
\begin{aligned}
&+2[(n+2)p_1^2-(n-10)p_2^2]\kappa_{2,1}-(n-1)(n-2)^2p_2^2\times\\
&\times[3-p_2^2(p_1^2)^{-1}]\widetilde{\kappa}-\frac{1}{3}(n-2)^2(p_1^2)^{-1}(p_1^2-p_2^2)^2\times\\
&\times[3\kappa_{2,1}+2(n-1)\widetilde{\kappa}]+2(n-2)m^2(p_1^2)^{-2}(5p_1^2-p_2^2)\times\\
&\times(p_1^2+p_2^2)[2\kappa_{2,1}-(n-2)\widetilde{\kappa}]\Big\}. \qquad (3.18)
\end{aligned}
$$

The functions A, B, C and F with the arguments $(p_2^2, 0, p_1^2)$ can be obtained from those with arguments $(0, p_1^2, p_2^2)$ by interchanging $p_1^2 \leftrightarrow p_2^2$. We remind that A, C and F are symmetric in the first two arguments, whereas B is antisymmetric. The function H is totally symmetric with respect to all the three arguments.

Considering the massless limit, we should substitute $\widetilde{\kappa} \Rightarrow 0$, and

$$
\varphi_3(p_1^2,0,0) \Rightarrow -\frac{2(n-3)\kappa_1}{(n-4)p_1^2}, \qquad \varphi_3(0,p_2^2,0) \Rightarrow -\frac{2(n-3)\kappa_2}{(n-4)p_2^2}\,. \qquad (3.19)
$$

Using these relations, we see that the results given in eqs. (3.17) and (3.18) [together with eqs. (3.2)–(3.4)] agree with those given in ref. [8, eqs. (4.15)–(4.23)]. We have also checked that in the massive case [8, eq. (4.24)] is satisfied by the one-loop expressions.

3.4 Zero-momentum limit, $p_3 = 0$

Putting $p_3 = 0$ ($p_1 = -p_2 \equiv p$, $p_1^2 = p_2^2 = p^2$), we get the three-gluon vertex eq. (2.2) in terms of three tensor structures [7, 8],

$$
\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(p,-p,0) = {}& 2g_{\mu_1\mu_2}p_{\mu_3}\left[A(p^2,p^2;0)+p^2C(p^2,p^2;0)\right]-2p_{\mu_1}p_{\mu_2}p_{\mu_3}C(p^2,p^2;0)-\\
&-\left(g_{\mu_1\mu_3}p_{\mu_2}+g_{\mu_2\mu_3}p_{\mu_1}\right)\left[A(0,p^2;p^2)-B(0,p^2;p^2)\right]. \qquad (3.20)
\end{aligned}
$$

Moreover, due to the relation

$$
A(0,p^2;p^2)-B(0,p^2;p^2)-A(p^2,p^2;0)=0\,, \qquad (3.21)
$$

the tensor structures in eq. (3.20) reduce to two independent ones [22, 8],

$$
\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(p,-p,0) = {}& \left(2g_{\mu_1\mu_2}p_{\mu_3}-g_{\mu_1\mu_3}p_{\mu_2}-g_{\mu_2\mu_3}p_{\mu_1}\right)A(p^2,p^2;0)+\\
&+2p_{\mu_3}\left(p^2g_{\mu_1\mu_2}-p_{\mu_1}p_{\mu_2}\right)C(p^2,p^2;0)\,. \qquad (3.22)
\end{aligned}
$$

This reduction is a corollary of the differential (zero-momentum) version of the WST identity. It is discussed in detail in Sec. III of ref. [23]. Note that the functions $T_i(p^2)$ used in refs. [22, 23] are related to the functions in eq. (3.22) as

$$
T_1(p^2) \leftrightarrow A(p^2,p^2;0), \quad T_2(p^2) \leftrightarrow -2p^2C(p^2,p^2;0)\,. \qquad (3.23)
$$

The results for these functions can be obtained from eqs. (3.2) and (3.4) by taking the limit $p_2^2 \to p_1^2 \equiv p^2$, using eq. (3.11). In this way, we obtain

$$
\begin{aligned}
A^{(1,q)}(p^2,p^2;0) &= \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{n-1}\left[\left(n-2+4\frac{m^2}{p^2}\right)\kappa_2(p^2) - 2(n-2)\frac{m^2}{p^2}\widetilde{\kappa}\right], \\
C^{(1,q)}(p^2,p^2;0) &= \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{1}{(n-1)(p^2-4m^2)}\left[(n-4)\left(n-2+4\frac{m^2}{p^2}\right)\kappa_2(p^2) + \right. \\
&\quad \left. +2\frac{m^2}{p^2}\left(n-4+12\frac{m^2}{p^2}\right)[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}]\right]. \qquad (3.24)
\end{aligned}
$$

In the massless limit, putting $\kappa_2(p^2) \Rightarrow \kappa_0(p^2) \equiv \kappa(p^2)$, $\widetilde{\kappa} \Rightarrow 0$, we see that the results given in eq. (3.24) agree with ref. [8, eqs. (4.32)–(4.34)].

3.5 Two gluons on shell, $p_1^2 = p_2^2 = 0$

In the limit $p_1^2 = p_2^2 = 0$ $[p_3^2 \equiv p^2$, $(p_1 p_2) = \frac{1}{2}p^2]$, the three-gluon vertex (2.2) involves seven tensor structures, and thus seven independent scalar functions $U_i(p^2)$ (see in refs. [7, 8]),

$$
\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\big|_{p_1^2=p_2^2=0} &= g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}U_1(p^2) + \left(g_{\mu_1\mu_3}p_{1\mu_2} - g_{\mu_2\mu_3}p_{2\mu_1}\right)U_2(p^2) + \\
&+ \left(g_{\mu_1\mu_3}p_{2\mu_2} - g_{\mu_2\mu_3}p_{1\mu_1}\right)U_3(p^2) + \\
&+ p_{1\mu_1}p_{2\mu_2}(p_1-p_2)_{\mu_3}U_4(p^2) + p_{1\mu_2}p_{2\mu_1}(p_1-p_2)_{\mu_3}U_5(p^2) + \\
&+ \left(p_{1\mu_1}p_{1\mu_2}p_{1\mu_3} - p_{2\mu_1}p_{2\mu_2}p_{2\mu_3}\right)U_6(p^2) + \\
&+ \left(p_{1\mu_1}p_{1\mu_2}p_{2\mu_3} - p_{2\mu_1}p_{2\mu_2}p_{1\mu_3}\right)U_7(p^2). \qquad (3.25)
\end{aligned}
$$

The functions $U_i(p^2)$ can be related to the functions A, B, C, F and H of eq. (2.2). The corresponding relations are given in [8, eqs. (4.59)–(4.65)].

The results for one-loop contributions to the scalar functions can be obtained in two ways. First, taking the limit $p_1^2 = p_2^2 \equiv p^2$ and then putting $p^2 = 0$. Secondly, starting from the results of section 3.3 and thereafter putting another momentum on shell. Both ways lead to the same results for quark-loop contributions to the U_i functions:

$$
\begin{aligned}
U_1^{(1,q)}(p^2) &= -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{(n-1)(n-2)}\left\{4(n-1)m^2\varphi_3(0,0,p^2) - n(n-3)\times \right. \\
&\quad \left. \times\kappa_2(p^2) + 4(n-2)\frac{m^2}{p^2}[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}]\right\}, \\
U_2^{(1,q)}(p^2) &= -\hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{4}{(n-1)}\left\{(n-2)\kappa_2(p^2) + 2\frac{m^2}{p^2}[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}]\right\}, \\
U_3^{(1,q)}(p^2) &= \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{1}{(n-1)(n-2)}\left\{8(n-1)m^2\varphi_3(0,0,p^2) + 4\kappa_2(p^2) + \right. \\
&\quad \left. +8(n-2)\frac{m^2}{p^2}[2\kappa_2(p^2)-(n-2)\widetilde{\kappa}] - (n-1)(n-2)^2\widetilde{\kappa}\right\},
\end{aligned}
$$

$$U_4^{(1,q)}(p^2) = \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{(n-1)(n-2)p^2}\Big\{16(n-1)m^2\varphi_3(0,0,p^2) + 2(n+2)\times$$
$$\times\kappa_2(p^2) + 12(n-2)\frac{m^2}{p^2}[2\kappa_2(p^2) - (n-2)\widetilde{\kappa}] - (n-1)(n-2)^2\widetilde{\kappa}\Big\}\,,$$
$$U_5^{(1,q)}(p^2) = \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{4}{(n-1)(n-2)p^2}\Big\{4(n-1)m^2\varphi_3(0,0,p^2) - (n-4)\times$$
$$\times\kappa_2(p^2) + 6(n-2)\frac{m^2}{p^2}[2\kappa_2(p^2) - (n-2)\widetilde{\kappa}]\Big\},$$
$$U_6^{(1,q)}(p^2) = \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{(n-1)p^2}\Big\{\left(n-2+\frac{4m^2}{p^2}\right)[2\kappa_2(p^2) - (n-2)\widetilde{\kappa}] +$$
$$+\frac{1}{3}(n-2)(n-4)\widetilde{\kappa}\Big\},$$
$$U_7^{(1,q)}(p^2) = \hat{N}_f T_R \frac{g^2\,\eta}{(4\pi)^{n/2}} \frac{2}{(n-1)(n-2)p^2}\Big\{8(n-1)m^2\varphi_3(0,0,p^2) - 2(n-4)\times$$
$$\times\kappa_2(p^2) + (n-2)\left(n-1+12\frac{m^2}{p^2}\right)[2\kappa_2(p^2) - (n-2)\widetilde{\kappa}]\Big\}\,. \qquad (3.26)$$

In the massless limit $m = 0$, we should put $\widetilde{\kappa} \Rightarrow 0$. In this way, we reproduce the corresponding results [8, (F8)–(F14)].

A useful check on the results (3.26) is to consider the limit $p^2 \to 0$, when the third gluon is also on shell. Since we have some p^2 in the denominators, we need to take a few terms of the expansions of $\kappa_2(p^2)$ and $\varphi_3(0,0,p^2)$ in p^2. The expansion of $\kappa_2(p^2)$ is given in eq. (3.13), while (cf. [18, eqs. (37), (38)])

$$\varphi_3(0,0,p^2) = \frac{(n-4)(n-2)}{8m^2}\widetilde{\kappa}\left[1 - \frac{n-6}{24}\frac{p^2}{m^2} + \mathcal{O}\left(\frac{(p^2)^2}{m^4}\right)\right]. \qquad (3.27)$$

In this way, we reproduce the results (3.9), (3.15) for the totally on-shell configuration.

4. Scalar three-point function

4.1 General off-shell case

We collect here some results for the scalar three-point integrals (2.3). General results for such integrals are given in [18][6], in terms of a triple hypergeometric series in the variables p_i^2/m^2.

[6]There is a misprint in a representation for the Φ_3 function given in the last line of [18, eq. (38)]: the arguments of the hypergeometric function should read $\frac{z_1}{4}, \frac{z_2}{4}, \frac{z_3}{4}$ (rather than z_1, z_2, z_3). The representation given in the second line of [18, eq. (38)], as well as the generalization of this result to the case of N-point diagrams given in [24, eq. (4.7)], are correct.

We are mainly interested in the case of unit powers of the denominators, $\nu_1 = \nu_2 = \nu_3 = 1$. We note that all integrals with higher integer powers of the propagators can be reduced to $J_3(1,1,1)$ and two-point integrals, by using a recurrence procedure [25] based on the integration-by-parts technique [26] (see also in [27]).

Transforming Feynman parametric integrals, we get

$$J_3(1,1,1) = -\mathrm{i}\pi^{n/2}\Gamma\left(3-\frac{n}{2}\right)\int\limits_0^\infty\int\limits_0^\infty \frac{\mathrm{d}\xi\ \mathrm{d}\eta}{(1+\xi+\eta)^{n-3}\left[m^2(1+\xi+\eta)^2-\xi p_1^2-\eta p_2^2-\xi\eta p_3^2\right]^{3-n/2}}\,. \tag{4.1}$$

In three dimensions ($n = 3$), these integrals can be evaluated in terms of elementary functions [28] (see also Sec. VA in [14]). In four dimensions, shifting the integration variables in (4.1), one can obtain the standard representation of the three-point function in terms of dilogarithms [29] (see also in [30]).

Another representation of the four-dimensional result, in terms of the Clausen function $\mathrm{Cl}_2(\theta) = \mathrm{Im}\left[\mathrm{Li}_2\left(e^{\mathrm{i}\theta}\right)\right]$ (see in [31]), can be derived using the geometrical approach of [14] (see also in [32]). Further details and explicit results for the general case can be found in Sec. VB of [14]. These results are related to those of ref. [29] by analytic continuation.

We would like to note that the approach of [14] also provides results valid in an arbitrary dimension $n = 4-2\varepsilon$, in terms of one-fold angular integrals of the type[7]

$$\frac{1}{2\varepsilon}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right] \tag{4.2}$$

(see also in [33, 34]). Using a simple substitution of variables, $\phi = \arctan(\sqrt{z}/\sin\eta)$, the results for these integrals can be expressed in terms of Appell's hypergeometric function F_1 of two variables, similar to those obtained by Tarasov [35] by using recurrence relations with respect to the space-time dimension.

The structure of singularities of the general three-point function, including the anomalous thresholds, was studied in ref. [36].

4.2 Symmetric case

In the completely symmetric case, we have $p_1^2 = p_2^2 = p_3^2 \equiv p^2$, $m_1 = m_2 = m_3 \equiv m$. Let us follow the geometrical approach of [14] to calculate the integral $J_3(1,1,1)$ in this case. The geometrical variables defined in [14, figure 6] in this symmetric configuration take the following values:

$$\tau_{12} = \tau_{23} = \tau_{31} \equiv \tau, \quad \cos\tau = 1-\frac{p^2}{2m^2}\,, \tag{4.3}$$

[7] For the general three-point function with different masses, we get six integrals of the type (4.2). In the case of equal masses, the number of different integrals reduces to three (cf. ref. [14, figure 7]).

$$D^{(3)} = (1-\cos\tau)^2(1+2\cos\tau) = \frac{(p^2)^2}{4m^4}\left(3-\frac{p^2}{m^2}\right), \tag{4.4}$$

$$\Lambda^{(3)} = \frac{3}{4}(p^2)^2, \quad m_0 = m^3\sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}} = m\sqrt{1-\frac{p^2}{3m^2}}, \tag{4.5}$$

$$\tau_{01} = \tau_{02} = \tau_{03} \equiv \tau_0, \quad \cos\tau_0 = \frac{m_0}{m} = \sqrt{1-\frac{p^2}{3m^2}}, \tag{4.6}$$

$$\varphi_{12} = \varphi_{23} = \varphi_{31} \equiv \varphi = \frac{2\pi}{3}, \tag{4.7}$$

$$\eta_{12} = \eta_{23} = \eta_{31} \equiv \eta, \quad \tan\eta = \cos\frac{\varphi}{2}\tan\tau_0 = \frac{1}{2}\tan\tau_0, \tag{4.8}$$

$$\tan^2\eta = \frac{p^2}{4(3m^2-p^2)}. \tag{4.9}$$

The latter quantity is positive for $0 < p^2 < 3m^2$, and negative otherwise.

Let us use [14, eqs. (5.16)–(5.17)], remembering that in the symmetric case the result should be multiplied by

$$-\frac{3\mathrm{i}\pi^2}{\sqrt{\Lambda^{(3)}}} = -\frac{2\mathrm{i}\pi^2\sqrt{3}}{p^2}. \tag{4.10}$$

From [14, eq. (5.16)] [cf. also eq. (4.2)] we get, for the case of four dimensions:

$$J_3(1,1,1)\big|_{p_i^2=p^2,\, n=4} = -\frac{2\mathrm{i}\pi^2\sqrt{3}}{p^2}\int\limits_0^{\pi/3}\mathrm{d}\phi\ \ln\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right). \tag{4.11}$$

In the region $0 < p^2 < 3m^2$ (when the momentum is timelike but its square does not exceed the anomalous threshold $3m^2$), we can directly use the geometrical result (5.17) from [14],

$$\begin{aligned} J_3(1,1,1)\big|_{p_i^2=p^2,\, n=4} = -\frac{\mathrm{i}\pi^2\sqrt{3}}{p^2}\Bigg[&\mathrm{Cl}_2\left(\frac{2\pi}{3}+\tau\right) + \mathrm{Cl}_2\left(\frac{2\pi}{3}-\tau\right) - 2\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) + \\ &+ \tau\ln\left(\frac{\sin(\pi/3+\tau/2)}{\sin(\pi/3-\tau/2)}\right)\Bigg], \end{aligned} \tag{4.12}$$

where $\mathrm{Cl}_2(\theta)$ is the Clausen function [31] [remember that $\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) = \frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$]. Note that there is a logarithmic singularity at the anomalous threshold $p^2 = 3m^2$ ($\tau = \frac{2\pi}{3}$). In the special cases $p^2 = m^2$ and $p^2 = 2m^2$ we get, respectively,

$$J_3(1,1,1)\big|_{p_i^2=m^2,\, n=4} = \frac{\mathrm{i}\pi^2}{m^2\sqrt{3}}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \pi\ln 2\right], \tag{4.13}$$

$$J_3(1,1,1)\big|_{p_i^2=2m^2,\, n=4} = \frac{\mathrm{i}\pi^2}{4m^2\sqrt{3}}\left[5\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - 3\pi\ln\left(2+\sqrt{3}\right)\right]. \tag{4.14}$$

To analytically continue the result (4.12) to other regions of interest, we can use the representation (4.11). We note that $\tan^2\eta < 0$ for $p^2 > 3m^2$ and for $p^2 < 0$.

JHEP08(2001)050

To describe the Euclidean region ($p^2 = -\mu^2 < 0$), as well as the region above the two-particle threshold ($p^2 > 4m^2$), it is convenient to introduce the angle τ' such that

$$\cos\tau' = \frac{6m^2 - p^2}{2(3m^2 - p^2)} = \frac{6m^2 + \mu^2}{2(3m^2 + \mu^2)} . \tag{4.15}$$

The argument of the logarithm in eq. (4.11) is always positive for $p^2 < 0$, and always negative for $3m^2 < p^2 < 4m^2$. In the region $p^2 > 4m^2$, the argument is positive for $0 < \phi < \frac{1}{2}(\pi - \tau')$ and negative for $\frac{1}{2}(\pi - \tau') < \phi < \frac{\pi}{3}$. This means that we obtain an imaginary part for $3m^2 < p^2 < 4m^2$ and for $p^2 > 4m^2$.

The occurring angular integrals can be calculated by using eqs. (33), (34), (36) and (38) on p. 308 of [31]. In this way, we arrive at the following results. For $3m^2 < p^2 < 4m^2$, we have

$$J_3(1,1,1)|_{p_i^2=p^2,\, n=4} = -\frac{\mathrm{i}\pi^2\sqrt{3}}{p^2}\left[\mathrm{Cl}_2\left(\frac{2\pi}{3}+\tau\right) + \mathrm{Cl}_2\left(\frac{2\pi}{3}-\tau\right) - 2\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) + \right.$$
$$\left. + \tau\ln\left(\frac{\sin(\tau/2+\pi/3)}{\sin(\tau/2-\pi/3)}\right) + \frac{2\mathrm{i}\pi^2}{3}\right]. \tag{4.16}$$

For $p^2 > 4m^2$, we have

$$J_3(1,1,1)|_{p_i^2=p^2,\, n=4} = -\frac{\mathrm{i}\pi^2\sqrt{3}}{p^2}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}+\tau'\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}-\tau'\right) - 2\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \right.$$
$$\left. + \mathrm{i}\pi\left(\tau' - \frac{\pi}{3}\right)\right]. \tag{4.17}$$

In particular, at $p^2 = 4m^2$ we get

$$J_3(1,1,1)|_{p_i^2=4m^2,\, n=4} = \frac{\mathrm{i}\pi^2}{2m^2\sqrt{3}}\left[5\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \mathrm{i}\pi^2\right] . \tag{4.18}$$

Finally, in the euclidean region, $p^2 = -\mu^2 < 0$, we get

$$J_3(1,1,1)|_{p_i^2=-\mu^2,\, n=4} = \frac{\mathrm{i}\pi^2\sqrt{3}}{\mu^2}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}+\tau'\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}-\tau'\right) - 2\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]. \tag{4.19}$$

The latter result gives the correct massless limit [3],

$$J_3(1,1,1)|_{p_i^2=-\mu^2,\, m=0,\, n=4} = -\frac{4\mathrm{i}\pi^2}{\mu^2\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) . \tag{4.20}$$

In figure 2 we show the function $\varphi_3(p^2,p^2,p^2)$, which in $n = 4$ dimensions is given by $J_3(1,1,1)/(\mathrm{i}\pi^2)$. The real part is seen to be singular at the anomalous threshold, $p^2 = 3m^2$, whereas it has a kink (the first derivative is discontinuous) at the normal (two-particle) threshold, $p^2 = 4m^2$. The imaginary part starts at a finite value for $p^2 = 3m^2$, and has a kink at $p^2 = 4m^2$.

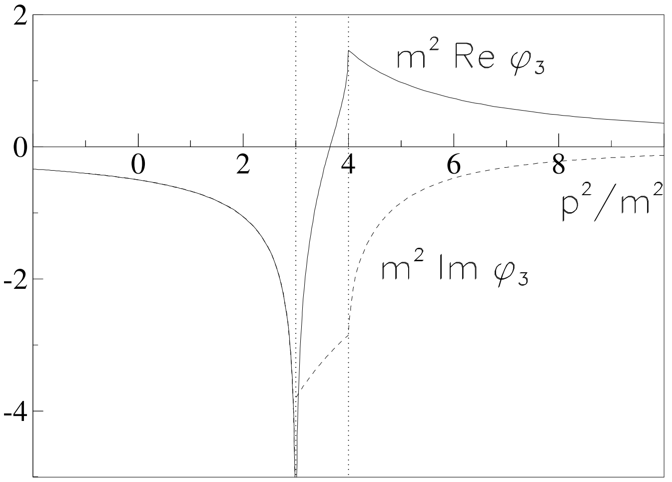

Figure 2: The integral $\varphi_3(p^2, p^2, p^2)$ in four dimensions, $n = 4$.

4.3 On-shell and zero-momentum cases

When all momenta squared vanish, we get a tadpole,

$$J_3(1,1,1)\big|_{p_1^2=p_2^2=p_3^2=0} = -\frac{1}{2}\mathrm{i}\pi^{n/2}(m^2)^{n/2-3}\Gamma(3-n/2)\,. \tag{4.21}$$

When two momenta are on shell, $p_1^2 = p_2^2 = 0$, the integral $J_3(1,1,1)$ can be presented in arbitrary dimension as (see, e.g. [18, eq. (40)]):

$$J_3(1,1,1)\big|_{p_1^2=p_2^2=0} = -\frac{1}{2}\mathrm{i}\pi^{n/2}(m^2)^{n/2-3}\Gamma(3-n/2)\;{}_3F_2\left(\begin{matrix}3-n/2,\ 1,\ 1\\ 3/2,\ 2\end{matrix}\;\middle|\;\frac{p_3^2}{4m^2}\right), \tag{4.22}$$

where ${}_PF_Q$ is the generalized hypergeometric function. A number of integral representations for this integral are given in [34, sec. 3.3], where also explicit results for some terms of the ε-expansion are presented. In four dimensions,

$${}_3F_2\left(\begin{matrix}1,\ 1,\ 1\\ 3/2,\ 2\end{matrix}\;\middle|\;z\right) = \begin{cases} z^{-1}\arcsin^2\sqrt{z}\,, & z \geq 0\,,\\ -z^{-1}\ln^2\left(\sqrt{1-z}+\sqrt{-z}\right), & z \leq 0\,. \end{cases} \tag{4.23}$$

This is the familiar result [37] for the case of a Higgs particle coupling to two massless vector particles, Hgg or $H\gamma\gamma$ (see also in [16]).

In the case when just one momentum is on shell, $p_3^2 = 0$, we can start from the general hypergeometric representation (see [18, eq. (37)]). Putting $p_3^2 = 0$ we get a double hypergeometric series. Then, denoting $j_1 + j_2 = j$ and summing over the remaining index, we arrive at

$$\begin{aligned} J_3(1,1,1)\big|_{p_3^2=0} &= -\frac{1}{2}\mathrm{i}\pi^{n/2}(m^2)^{n/2-3}\Gamma(3-n/2)\,\frac{1}{p_1^2-p_2^2}\times \\ &\times\left[p_1^2\;{}_3F_2\left(\begin{matrix}3-n/2,1,1\\ 3/2,\ 2\end{matrix}\;\middle|\;\frac{p_1^2}{4m^2}\right) - p_2^2\;{}_3F_2\left(\begin{matrix}3-n/2,1,1\\ 3/2,\ 2\end{matrix}\;\middle|\;\frac{p_2^2}{4m^2}\right)\right], \end{aligned} \tag{4.24}$$

JHEP08(2001)050

or

$$J_3(1,1,1)|_{p_3^2=0} = \frac{1}{p_1^2 - p_2^2}\left[p_1^2 \; J_3(1,1,1)|_{p_2^2=p_3^2=0} - p_2^2 \; J_3(1,1,1)|_{p_1^2=p_3^2=0}\right], \qquad (4.25)$$

i.e., it is just a linear combination of the integrals (4.22) with two legs on shell. In particular, eq. (4.25) means that all results of sec. 3.3 of [34] are applicable to the case of two legs off shell, too. For $n = 4$, eq. (4.25) yields a combination of elementary functions (4.23) — this result is known from the calculation of the one-loop $H \to Z\gamma$ vertex [38] (see also in [16]).

In the case when $p_3 = 0$ ($p_1 = -p_2 \equiv p$), the three-point integral J_3 effectively becomes a two-point integral, with one of the massive denominators to the second power. Formally, we can write

$$J_3(1,1,1)|_{p_3=0} = J_3(0,2,1)\,. \qquad (4.26)$$

Using integration by parts [26] (see also [39, eq. (A.17)]), $J_3(0,2,1)$ can be reduced to $J_3(0,1,1)$ and a tadpole integral. In this way, we obtain

$$J_3(1,1,1)|_{p_3=0} = \frac{1}{2(4m^2 - p^2)}\left[2(n-3)J_3(0,1,1) - \frac{1}{m^2}(n-2)J_3(0,0,1)\right], \qquad (4.27)$$

or, in the notation of eqs. (2.4)–(2.7),

$$\varphi_3(p^2, p^2, 0) = \frac{1}{2(4m^2 - p^2)}[2(n-3)\kappa_2(p^2) - (n-2)\widetilde{\kappa}]\,. \qquad (4.28)$$

5. Conclusions

We have obtained general results for the quark-loop contributions to the three-gluon vertex in an arbitrary dimension. For the general off-shell case, the decomposition of Ball and Chiu [6] was used. The general off-shell case, as well as all on-shell limits of interest have been considered. The obtained results satisfy the corresponding Ward-Slavnov-Taylor identity (2.16). We have also presented some results for the corresponding three-point integrals.

This calculation completes the investigation of the one-loop three-gluon vertex in an arbitrary dimension, which was initiated in [8]. This could be a valuable element in two-loop (and higher) calculations. Moreover, together with ref. [10] the calculation completes the study of one-loop three-point vertices in QCD, in an arbitrary covariant gauge.

A similar study of the two-loop-order corrections to the three-gluon vertex (and other QCD vertices) requires more involved techniques. We note that for some special configurations (for massless quarks) two-loop results for the three-gluon vertex are already available, in particular, for the zero-momentum case [22, 23] and the $p_1^2 = p_2^2 = 0$ case [40]. A numerical approach to the symmetric case has been developed in [41]. Moreover, the three-loop results in the zero-momentum case are available in [42].

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JHEP08(2001)050

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